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GENERALIZED TWO-DIMENSIONAL ABELIAN GAUGE THEORIES AND CONFINEMENT

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We consider the Abelian generalization of QED_2 to include SU(M) flavour and "diagonal" SU(N) colour. The operator solutions and confinement aspects of these models are discussed in detail for the case of massless and massive fermions. For a non-vanishing fermion mass one finds confinement of "quarks" except for some special "θ-worlds".

1. Introduction

There has been much interest recently in Quantum Field Theories in two-dimensional space-time, since they provide a very instructive framework for studying non-perturbative aspects. One such aspect of particular interest has been the question of quark confinement. In two-dimensional gauge theories, confinement may appear to be an automatic consequence of Gauss' law and the dimensionality of space-time. This is not necessarily so since Gauss' law here only *forbids* the existence of sectors carrying quantum numbers *coupling to the gauge field* [1]. For QED_2 this implies the absence of states carrying electric charge. This could either mean that the quark charge has been screened or that quarks have been permanently bound into hadrons. In the bound-state picture, the mass of the "quark" is expected to play a fundamental role, since it will help to set the scale of hadronic interactions and, hence, of hadronic size.

It is generally believed that "vortex formation" will be an essential ingredient of a confining theory [2]. Taking the existence of such vortices for granted, one could take up the problem at this point and ask in which way confinement will manifest itself. It appears thus reasonable to pursue this question in the context of models in

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$1+1$ dimensions where the existence of such vortices is an automatic consequence of the dimensionality of space-time.

Although the question of screening *versus* confinement will ultimately be a question of a more detailed dynamical understanding, a good indication for what actually is happening could already be obtained if the fundamental fermion field involves additional quantum numbers which cannot be screened. This would be triality in the case of $U(1)_G \times SU(M)_F$ (G = gauge group, F = flavour group) and electric charge in the case of $U(1)_F \times SU(N)_G$.

The particular case of SU(2) flavour has already been discussed in ref. [3]. Here we consider generalizations to the above symmetry groups. However, since we shall be interested in arriving at a non-perturbative picture of the spectrum of states, we restrict ourselves to the maximal Abelian (diagonal) subgroup (torus) $SU(N)_D$ of $SU(N)_G$ colour. In the case of massless fermions, exact solutions can then be constructed following the methods of ref. [4]. The really interesting case is the one where the fermions do have a mass [5]. In that case we find a clear signal for confinement of triality, except for some special θ-worlds where the confinement picture is replaced by that of a screened quark. These θ-worlds together with $\theta=0$ are precisely the P, T invariant points in the set of all θ-vacua.

The existence of such screened exotic states for a discrete set of θ-worlds with massive fermions show that screening may play an important role also in situations where typical hadronic mass scales are involved. Such states would presumably only show up if, as will be done here, the vacuum polarization effects are taken into account. This is not the case for the Wilson-loop criterion [6] which neglects these polarization effects.

The present article is intended to give a reasonably self-contained discussion of various aspects of generalized gauge theories in $1+1$ dimension with coloured and flavoured "quarks". The paper is divided into three main sections in which we discuss separately the generalizations corresponding to a $SU(M)_F$ flavour, $SU(N)_{D,G}$ colour, and $SU(M)_F \times SU(N)_{D,G}$ gauge symmetry group, respectively. In each case we discuss the solutions and the implied vacuum structure, clustering and confinement properties separately for the case of massless and massive fermions. As we shall see, the mass of the fermion will play a fundamental role in confining states of non-zero triality.

In order to facilitate reading of this paper we shall outline in some detail the main results.

In sect. 2 we generalize the well-known results of QED_2 to the case of $U(1)_G \times SU(M)_F$ flavour, where the U(1) gauge field only couples to the charge of the "quarks". Charged sectors are absent as required by Gauss' law [1]. The structure of the θ-vacuum is found to be the same as in QED_2, but we have now an additional (gauge invariant) operator $\mathscr{F}_\lambda(x)$ (eq. (2.17a)) at our disposal which carries *fundamental* flavour and is thus a candidate for creating physical states carrying quark quantum numbers. For zero-mass fermions $\mathscr{F}_\lambda(x)$ does indeed create such quark-like states, indicating that the absence of charge sectors merely reflects screening of the quark charge as induced by the electro-magnetic interaction. For massive fermions this is no longer found to be true, except for some isolated θ-worlds: for $\theta \neq \pm \pi/M$ only "non-exotic" states remain in the spectrum. Since flavour cannot be screened by the electromagnetic interaction we conclude that the quarks must have been permanently bound into non-exotic (hadronic) states. This turns out to be entirely in agreement with a semiclassical dynamical picture developed recently [17]. For $\theta = \pm \pi/M$ also states carrying fundamental flavour make their appearance thus indicating that charge-screened quarks are liberated.

In sect. 3 we then consider the generalization of QED_2 to the maximal Abelian subgroup $SU(N)_D$ of the $SU(N)_G$ gauge group, the torus of $SU(N)$. Coloured sectors are again absent as required by the generalized Gauss' law in $1+1$ dimensions. This could mean screening or confinement. In the absence of a U(1) gauge field the "quark" charge cannot be screened. Hence we use the electromagnetic charge to distinguish between the above two possibilities. The charge now plays the role of flavour in the previous model, and correspondingly the soliton-like operator $S(x)$ in eq. (3.17c) now takes up the role of the flavour operator $\mathscr{F}_\lambda(x)$. Since $S(x)$ is gauge invariant, it is a candidate for creating states carrying the elementary quark charge, and thus serves as a probe for confinement. For zero-mass fermions $S(x)$ is indeed found to create states carrying one unit of charge, thus indicating that screening rather than confinement is responsible for the absence of colored states in this case. For massive fermions this is no longer true except for some isolated values of the now $N-1$ independent θ-angles characterizing the gauge vacuum: the spectrum only contains the usual non-exotic (hadronic) states except for $\theta^a = \pm \pi/N$, where quarks are again liberated.

In sect. 4 we then combine the above symmetry groups into $SU(M)_F \times SU(N)_{D,G}$. The flavour and "soliton" operators now make jointly their appearance. The screening (massless case) and confining (massive case) features of the previous two models persist except that the now different vacuum structure exhibits an additional "selectivity" of the isolated θ-worlds in which "quarks" are liberated.

We conclude in sects. 5 and 6 with some remarks on the relevance of these results to QCD_2 and a brief summary.

Throughout this paper we follow the conventions of ref. [4].

2. QED_2 with flavour

2.1. The massless model

To begin with we consider the simple extension of QED_2 to the case of M massless fermions transforming under the group $SU(M)$ of flavour and the gauge group U(1). The corresponding equations of motion are

$$i\gamma^\mu \partial_\mu \psi_\lambda(x) + \tfrac{1}{2}e \lim_{\epsilon\to 0} \gamma^\mu \{A_\mu(x+\epsilon)\psi_\lambda(x) + \psi_\lambda(x) A_\mu(x-\epsilon)\} = 0\,, \tag{2.1a}$$

$$\partial_\mu F^{\mu\nu}(x) + eJ^\nu(x) = 0\,, \tag{2.1b}$$

where λ is a flavour index,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,,$$

is the field-strength tensor, and J^μ is the electromagnetic current

$$J^\mu(x) = \sum_{\lambda=1}^{M} N[\bar{\psi}_\lambda(x)\gamma^\mu\psi_\lambda(x)]\,, \tag{2.2}$$

where the normal product is defined by the usual limiting procedure [7].

The solution to eq. (2.1a) in the "Schwinger gauge" [7] will be of the form

$$\psi_\lambda(x) = :\exp\{i\gamma^5[\alpha\tilde{\Sigma}(x) + \beta\tilde{\eta}(x)]\}:\,\psi_\lambda^{(0)}(x)\,, \tag{2.3}$$

with the identification

$$A_\mu(x) = -\frac{1}{e}\epsilon_{\mu\nu}\partial^\nu(\alpha\tilde{\Sigma}(x) + \beta\tilde{\eta}(x)) \tag{2.4}$$

for the gauge field, and $\tilde{\eta}$ a zero-mass (gauge) excitation. The free, massless canonical Dirac field $\psi_\lambda^{(0)}(x)$ can conveniently be written in the bosonized form [8]

$$\psi_\lambda^{(0)}(x) = \left(\frac{\mu}{2\pi}\right)^{1/2} \mathrm{e}^{-i\pi\gamma^5/4}:\exp\left[i\sqrt{\pi}\left\{\gamma^5\tilde{\varphi}_\lambda(x) + \int_{x^1}^{\infty}\partial_0\tilde{\varphi}_\lambda(x^0, z^1)\,\mathrm{d}z^1\right\}\right]:\,, \tag{2.5}$$

where the fields $\tilde{\varphi}_\lambda$ are the potentials associated with the M free, conserved currents

$$j_\lambda^\mu(x) = :\bar{\psi}_\lambda^{(0)}(x)\gamma^\mu\psi_\lambda^{(0)}(x): = -\frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu\tilde{\varphi}_\lambda\,, \tag{2.6}$$

and μ is an arbitrary parameter introduced by the infrared regularization of the zero-mass fields φ_λ.

The current (2.2) is calculated to be

$$J^\mu(x) = -\alpha\frac{M}{\pi}\epsilon^{\mu\nu}\partial_\nu\tilde{\Sigma}(x) + L^\mu(x)\,, \tag{2.7a}$$

with

$$L^\mu(x) = -\sqrt{\frac{M}{\pi}}\,\partial^\mu\left(\varphi + \sqrt{\frac{M}{\pi}}\beta\eta\right), \tag{2.7b}$$

where φ is the canonical free field defined by

$$\varphi = \frac{1}{\sqrt{M}}\sum_\lambda \varphi_\lambda\,, \tag{2.8}$$

and we have used

$$\epsilon_{\mu\nu}\partial^{\nu}\tilde{\eta}=\partial_{\mu}\eta\,,\qquad \epsilon_{\mu\nu}\partial^{\nu}\tilde{\varphi}=\partial_{\mu}\varphi\,, \tag{2.9}$$

Due to the presence of the longitudinal current L^{μ} the Maxwell equations (2.16) are only satisfied on the physical subspace defined by

$$\langle\Phi|L^{\mu}(x)|\Psi\rangle=0\,,\qquad |\Phi\rangle,|\Psi\rangle\in\mathscr{H}_{\mathrm{phys}}\,, \tag{2.10}$$

with $\tilde{\Sigma}$ satisfying the equation of motion

$$\left(\Box+\frac{Me^{2}}{\pi}\right)\tilde{\Sigma}(x)=0\,. \tag{2.11}$$

Condition (2.10) implies that $L^{\mu}(x)$, applied to the Fock vacuum, generates states of zero norm; hence, η must be a canonical, free field, quantized with indefinite metric. This fixes the constant β to be $\beta=\sqrt{\pi/M}$. The constant α is chosen to be $\alpha=\sqrt{\pi/M}$ in order for $\psi_{\lambda}(x)$ to approach the canonical free fermion field when the interaction is turned off. This also implies a canonical short-distance behaviour for the gauge invariant Green functions of the interacting field.

The SU(M) currents are just the free fermion currents. The $M-1$ diagonal ones are conveniently written in terms of $M-1$ canonical, massless fields $\tilde{\phi}^{i_{\mathrm{D}}}$,

$$J_{\mu}^{i_{\mathrm{D}}}(x)=N[\bar{\psi}(x)\gamma_{\mu}\tfrac{1}{2}\lambda^{i_{\mathrm{D}}}\psi(x)]=-\frac{1}{\sqrt{2\pi}}\epsilon e\mu\nu\partial^{\nu}\tilde{\phi}^{i_{\mathrm{D}}}(x)\,, \tag{2.12a}$$

$$J_{\mu}^{5i_{\mathrm{D}}}(x)=N[\bar{\psi}(x)\gamma^{5}\gamma_{\mu}\tfrac{1}{2}\lambda^{i_{\mathrm{D}}}\psi(x)]=-\frac{1}{\sqrt{2\pi}}\partial_{\mu}\tilde{\phi}^{i_{\mathrm{D}}}(x)\,, \tag{2.12b}$$

where

$$\mathrm{tr}\,(\lambda^{i},\lambda^{j})=2\delta^{ij}\,, \tag{2.13}$$

and the "potentials" $\phi^{i_{\mathrm{D}}}$ are related to those in eq. (2.6) by

$$\tilde{\varphi}_{\mathrm{f}}=\frac{1}{\sqrt{M}}\tilde{\varphi}+\sqrt{\frac{1}{2}}\sum_{i_{\mathrm{D}}}\lambda_{\mathrm{ff}}^{i_{\mathrm{D}}}\tilde{\phi}^{i_{\mathrm{D}}}\,. \tag{2.14}$$

The physical Hilbert space is generated by the application of polynomials of all the (gauge-invariant) operators commuting with $L_{\mu}(x)$, such as $F_{\mu\nu}$, J_{μ}, L_{μ}, $\hat{\psi}_{\lambda}$, where $\hat{\psi}_{\lambda}$ is obtained from the field (2.3) by performing the operator gauge transformation

$$\psi_{\lambda}(x)\to\,:\exp\left[i\sqrt{\frac{\pi}{M}}\int_{x^{1}}^{\infty}\mathrm{d}y^{1}\partial_{0}\tilde{\eta}\right]\psi_{\lambda}(x):\,\equiv\hat{\psi}_{\lambda}(x)\,, \tag{2.15a}$$

$$A_{\mu}(x)\to A_{\mu}(x)+\frac{1}{e}\sqrt{\frac{\pi}{M}}\partial_{\mu}\eta(x)\equiv\hat{A}_{\mu}(x)\,. \tag{2.15b}$$

In this "physical" gauge, $\hat{\psi}_\lambda$ is conveniently factorized in the following form:

$$\hat{\psi}_\lambda(x)=\left(\frac{\mu}{2\pi}\right)^{1/2}\exp\left[-\tfrac{1}{4}i\pi\gamma^5\right]:\exp\left\{i\sqrt{\frac{\pi}{M}}\gamma^5\tilde{\Sigma}(x)\right\}:\mathscr{F}_\lambda(x)\sigma, \tag{2.16}$$

where

$$\mathscr{F}_\lambda(x)=:\exp\left[i\sqrt{\pi}\gamma^5\left(\tilde{\varphi}_\lambda(x)-\frac{1}{\sqrt{M}}\tilde{\varphi}(x)\right)+i\sqrt{\pi}\int_{x^1}^{\infty}\mathrm{d}y^1\,\partial_0\left(\tilde{\varphi}_\lambda-\frac{1}{\sqrt{M}}\tilde{\varphi}\right)\right]:, \tag{2.17a}$$

$$\sigma=\exp\left[i\sqrt{\frac{\pi}{M}}\left\{\gamma^5\left(\tilde{\varphi}(x)+\tilde{\eta}(x)\right)+\int_{x^1}^{\infty}\mathrm{d}y^1\partial_0\left(\tilde{\varphi}+\tilde{\eta}\right)\right\}\right]. \tag{2.17b}$$

σ is an operator with scale dimension zero.

2.2. *Properties of the massless model*

The general properties of the model are similar to those of QED_2 without flavour. Nevertheless, there are some special features which will be exhibited in the course of the following discussion.

2.2.1. Vacuum structure and clustering. The operator σ_α(α = Lorentz index) in eq. (2.17) commutes with all the observables of the theory. On the physical subspace defined by (2.10) it acts as a constant operator which merely carries the bare-charge and chiral selection rules

$$[q,\sigma_\alpha]=-\sigma_\alpha,\qquad [Q_5,\sigma_\alpha]=-\gamma^5_{\alpha\alpha}\sigma_\alpha, \tag{2.18}$$

where q and $\tilde{Q}_5$ are the charges associated with the free U(1) current and the gauge-variant (in our case free) axial vector current, respectively. As in the case of the Schwinger model, we generate an infinite set of vacuum states [4] by repeated application of σ_α on the Fock vacuum.

$$|n_1,n_2\rangle=\sigma_1^{n_1}\sigma_2^{n_2}|0\rangle. \tag{2.19}$$

As is well-known, this vacuum degeneracy implies a violation of clustering. However, unlike the case of U[1] QED_2, this violation does not occur for the gauge-invariant fermion two-point function, since the operator J_λ in (2.16) carries flavour and SU(M) chirality

$$\begin{aligned}&[Q^{i_D},\mathscr{F}_f(x)]=-\tfrac{1}{2}\lambda^{i_D}_{ff}\mathscr{F}_f(x),\\&[Q_5^{i_D},\mathscr{F}_f(x)]=-\tfrac{1}{2}\lambda^{i_D}_{ff}\gamma^5\mathscr{F}_f(x),\end{aligned} \tag{2.20}$$

whereas the vacuum is flavour neutral and only carries U[1] chirality:

$$[Q^{i_D}, \sigma_\alpha] = [Q_5^{i_D}, \sigma_\alpha] = 0\,. \tag{2.21}$$

Here Q^{i_D} and $Q_5^{i_D}$ are the charges associated with the SU(M) currents (2.12). It follows from (2.21) that violation of clustering can at best occur for operators which are $\mathrm{SU}(M)_\mathrm{L} \times \mathrm{SU}(M)_\mathrm{R}$ singlets: if such operators are expressable as a sum of non-singlet operators, there will be no violation of clustering; an example is provided by the chiral density $J(x) = \sum J_\lambda(x)$ where $J_\lambda = \bar{\psi}_\lambda \frac{1}{2}(1+\gamma^5)\psi_\lambda$ carries 2 units of chirality:

$$\langle 0|J^\dagger(x)J(x+\xi)|0\rangle \underset{|\xi|\to\infty}{\sim} [\xi^2]^{(1/M)-1}\,. \tag{2.22}$$

On the other hand, the singlet operator constructed from products of J_λ does violate the cluster decomposition:

$$\lim_{|\xi|\to\infty} \langle 0| \prod_{\lambda=1}^{n} J_\lambda^\dagger(x_\lambda) \prod_{\lambda'=1}^{n} J_{\lambda'}(x_{\lambda'}+\xi)|0\rangle$$
$$= \begin{cases} 0 \text{ for } n \neq mM, \\ |\langle 0| \prod_{\lambda=1}^{mM} J_\lambda^\dagger(x_\lambda)\,|-mM, mM\rangle|^2 \neq 0\,. \end{cases} \tag{2.23}$$

The cluster decomposition is restored with respect to the physical vacuum obtained in the usual way by considering the coherent superposition

$$|\theta_1, \theta_2\rangle = \frac{1}{2\pi} \sum_{n_1,n_2} \mathrm{e}^{-in_1\theta_1 - in_2\theta_2}|n_1, n_2\rangle\,, \tag{2.24}$$

which also provides an irreducible representation for the observables

$$\sigma_\alpha|\theta_1, \theta_2\rangle = \mathrm{e}^{i\theta\alpha}|\theta_1, \theta_2\rangle\,. \tag{2.25}$$

In the Euclidean functional integral approach to this model, these rules for obtaining the $\theta = \theta_1 - \theta_2$ chiral vacuum state directly involve the use of the generic (i.e., arbitrary external $\mathcal{A}$) Atiyah-Singer zero-energy eigenstates [9]. A similar topological functional understanding for the origin of $\theta_\mathrm{C} = \theta_1 + \theta_2$ has not yet been achieved [9].

2.2.2. Gauge transformation and topology. The (pseudo) unitary operator [10]

$$T[\Lambda] = \mathrm{e}^{i(M/\pi)Q[\Lambda]}\,, \tag{2.26}$$

$$Q[\Lambda] = \int_{y^0=x^0} \mathrm{d}y^1 \{(\tilde{\varphi}(y) + \tilde{\eta}(y))\partial_1\Lambda(y) - (\varphi(y) + \eta(y))\partial_0\Lambda(y)\}\,,$$

with

$$\Box\Lambda = 0\,, \tag{2.27}$$

induces the c-number gauge transformation

$$T[\Lambda]\psi_\lambda(x)T^{-1}[\Lambda]=e^{i\Lambda(x)}\psi_\lambda(x)\,,$$
$$T[\Lambda]A_\mu(x)T^{-1}[\Lambda]=A_\mu(x)+(1/e)\partial_\mu\Lambda(x)\,. \tag{2.28}$$

For $\Lambda(x^0,x^1)$ satisfying in addition to (2.27) the boundary conditions

$$\Lambda_{1/M}(0,-\infty)=0\,,\qquad \Lambda_{1/M}(0,\infty)=2\pi/M\,, \tag{2.29}$$

one has

$$T[\Lambda_{1/M}]=\sigma_1^+\sigma_2, \qquad \text{on } \mathscr{H}_{\text{phys}}\Lambda\,. \tag{2.30}$$

As in ref. [10], one can establish a similar relation for the operator σ_α itself:

$$T[\tilde{\Lambda}_{\pm 1/2M}]=\sigma_{1,2}, \qquad \text{on } \mathscr{H}_{\text{phys}}\Lambda\,. \tag{2.31}$$

This shows that the vacuum states (2.19) carry a topological quantum number $\nu=(n_1-n_2)/2M=(1/2M)\times\text{chirality}$.

The role of this topological number, as well as the violation of clustering discussed previously can also be understood from the functional point of view, following the arguments of ref. [11]. In particular one can use the cluster violation in order to obtain Feynman path representations for matrix elements of operators connecting different topological vacua, following the methods of ref. [11]. Thus for the example of eq.(2.23), the limit will be controlled by the induced instanton [12]

$$A_\mu^{\text{cl}}(z)=\frac{2\pi}{Me}\epsilon_{\lambda\mu}\frac{\partial}{\partial z_\lambda}\sum_{i=1}^{n}(\mathscr{D}(x_i-z)-\mathscr{D}(x_i+\xi-z))$$
$$=A_\mu^{[-n/M]}(z)+A_\mu^{[n/M]}(z)\,,$$

where the superscript refers to the Chern number of the field configuration, and

$$\mathscr{D}(z)=\Delta(z;0)-\Delta(z;Me^2/\pi)\,,$$

$\Delta(z,m)$ being the two-point function corresponding to a free scalar field of mass m.

The limit in eq. (2.23) is controlled by the implicit ξ dependence of $A_\mu^{[n/M]}(z)$ which, for $n\neq$ multiple of M, is insufficient to compensate the power-like fall-off associated with the free-fermion Green functions [11]* . The non-vanishing matrix element on the right-hand side of eq. (2.23) is given in terms of the induced instanton configuration $A_\mu^{[-m]}$ carrying winding number $-m$.

2.3. The massive model

From the preceding analysis we conclude that the indefinite metric formulation of massless QED_2 with SU(M) flavour corresponds to the following Hamiltonian

* See also ref. [9].

density:

$$\mathcal{H}_0 = \tfrac{1}{2}\sum_{\lambda=1}^{M} :(\partial_0\tilde{\varphi}_\lambda)^2 + (\partial_1\tilde{\varphi}_\lambda)^2: + \tfrac{1}{2}:(\partial_0\tilde{\Sigma})^2 + (\partial_1\tilde{\Sigma})^2 + \frac{Me^2}{\pi}\tilde{\Sigma}^2: \\ -\tfrac{1}{2}:(\partial_0\tilde{\eta})^2 + (\partial_1\tilde{\eta})^2: . \tag{2.32}$$

We now consider the effect of introducing a fermion mass. The Hamiltonian (2.32) is then replaced by

$$\mathcal{H}(x) = \mathcal{H}_0(x) + m\sum_{\lambda=1}^{M} :\bar{\psi}_\lambda(x)\psi_\lambda(x): . \tag{2.33}$$

We again make an ansatz of the form (2.3) and (2.5) [13]:

$$\psi_\lambda(x) = \left(\frac{\mu}{2\pi}\right)^{1/2} e^{-i\pi\gamma^5/4} : \exp[i\gamma^5\{\alpha\tilde{\Sigma}(x) + \beta\tilde{\eta}(x)\}]: \\ \times : \exp\left[i\gamma^5\delta\tilde{\varphi}_\lambda(x) + i\frac{\pi}{\delta}\int_{x^1}^{\infty} dz^1\partial_0\tilde{\varphi}_\lambda\right]: . \tag{(2.34)}$$

However, the bosonic fields will no longer satisfy free-field equations. We shall nevertheless assume that these fields exhibit a free-field behaviour at short distances. This corresponds to introducing a mass perturbation which does not destroy the asymptotically free behaviour of the theory. Then, the mass operator $\mathcal{M}(x)$ defined in terms of the short-distance limit

$$\sum_{\lambda=1}^{N} \{\psi_1^{\lambda\dagger}(x+\xi)\psi_2^\lambda(x) + \psi_2^{\lambda\dagger}(x+\xi)\psi_1^\lambda(x)\}_{\xi\to 0} = (\mu^2\xi^2)^{1-\dim[\bar{\psi}\psi]}\mathcal{M}(x) ,$$

is given by

$$\mathcal{M}(x) = -\frac{\mu}{\pi}\sum_{\lambda=1}^{M} :\cos\{2(\alpha\tilde{\Sigma}(x) + \beta\tilde{\eta}(x) + \delta\tilde{\varphi}_\lambda(x))\}: , \tag{2.35}$$

and has the scale dimension

$$\dim[\bar{\psi}\psi] = \frac{\delta^2}{\pi} + \frac{\alpha^2 - \beta^2}{\pi} . \tag{2.36}$$

The Hamiltonian corresponding to the density (2.33) leads to the coupled equations of motion:

$$\left(\Box + \frac{Me^2}{\delta^2}\right)\tilde{\Sigma}(x) + 2\alpha\frac{m\mu}{\pi}\sum_{N=1}^{M} :\sin\{2(\alpha\tilde{\Sigma}(x) + \beta\tilde{\eta}(x) + \delta\tilde{\varphi}_\lambda(x))\}: = 0 , \tag{2.37a}$$

$$\Box\tilde{\eta}(x) - 2\beta\frac{m\mu}{\pi}\sum_{\lambda=1}^{M} :\sin\{2(\alpha\tilde{\Sigma}(x) + \beta\tilde{\eta}(x) + \delta\tilde{\varphi}_\lambda(x))\}: = 0 , \tag{2.37b}$$

$$\Box\tilde{\varphi}_\lambda(x)+2\delta\frac{m\mu}{\pi}:\sin\{2(\alpha\tilde{\Sigma}(x)+\beta\tilde{\eta}(x)+\delta\tilde{\varphi}_\lambda(x))\}:=0\,, \tag{2.37c}$$

$$\Box(\tilde{\eta}(x)+\tilde{\varphi}(x))+2\left(\frac{\delta}{\sqrt{M}}-\beta\right)\frac{m\mu}{\pi}\sum_{\lambda=1}^{M}:\sin\{2(\alpha\tilde{\Sigma}(x)+\beta\tilde{\eta}(x)+\delta\tilde{\varphi}_\lambda(x))\}:=0\,. \tag{2.37d}$$

Using the methods of ref. [14], one obtains, upon applying the Dirac operator on the smeared fermion field,

$$\begin{aligned} i\gamma^\mu\partial_\mu\psi^\lambda(x) = &-\tfrac{1}{2}e\lim_{s_f\to 0}\gamma^\mu\{A_\mu(x,f)\psi^\lambda(x,f)+\psi^\lambda(x,f)A_\mu(x,f)\} \\ &+\tfrac{1}{2}\left(\delta-\frac{\pi}{\delta}\right)\lim_{s_f\to 0}\gamma^\mu\{j_\mu^{\lambda\,\mathrm{Th}}(x,f)\psi^\lambda(x,f)+\psi^\lambda(x,f)j_\mu^{\lambda\,\mathrm{Th}}(x,f)\} \\ &-\gamma^0 m\int_{-\infty}^{+\infty}[\mathcal{M}(z),\psi^\lambda(x)]_{\mathrm{ET}}\,\mathrm{d}z^1\,, \end{aligned} \tag{2.38}$$

where A_μ is given by

$$A_\mu(x)=-\frac{1}{e}\epsilon_{\mu\nu}\partial^\nu\{\alpha\tilde{\Sigma}(x)+\tilde{\eta}(x)\}\,. \tag{2.39}$$

The second term on the right-hand side of eq. (2.38) corresponds to a Thirring-like coupling with

$$j_\mu^{\lambda\,\mathrm{Th}}(x)=-\epsilon_{\mu\nu}\partial^\nu\tilde{\varphi}_\lambda(x)\,. \tag{2.40}$$

To prevent the appearance of this coupling we must require

$$\delta=\sqrt{\pi}\,. \tag{2.41}$$

The other term in (2.38) represents the mass contribution; it will be given by

$$\gamma^0_{\alpha\beta}\int_{-\infty}^{+\infty}[\mathcal{M}(z),\psi^\lambda(x)]_{\mathrm{ET}}\,\mathrm{d}z^1=\psi^\lambda_\alpha(x)\,, \tag{2.42}$$

provided the scale dimension (2.36) of the mass operator has the canonical value. This is indeed the case and follows from the requirement that Maxwell's equations be satisfied on the gauge-invariant (physical) subspace, as we now show.

The gauge-invariant current obtained by the usual limiting procedure is given by

$$J_\mu(x)=-\frac{M\alpha}{\pi}\epsilon_{\mu\nu}\partial^\nu\tilde{\Sigma}(x)+L_\mu(x)\,, \tag{2.43}$$

with

$$L_\mu(x) = -\epsilon_{\mu\nu}\partial^\nu\left\{\sqrt{\frac{M}{\pi}}\,\tilde{\varphi}(x) + \frac{M\beta}{\pi}\,\tilde{\eta}(x)\right\}, \tag{2.44}$$

where we have used (2.41).

For the choice $\beta = \sqrt{\pi/M}$, L_μ becomes a purely longitudinal zero-mass field of zero norm, as can be seen from the equations of motion (2.37). On the gauge-invariant subspace defined by (2.10) we thus require

$$\begin{aligned}&\langle\Phi|\partial_\mu F^{\mu\nu}(x) + eJ^\nu(x)\,|\psi\rangle\\ &= \frac{2}{e}\frac{m\mu}{\pi}\langle\Phi|(\alpha^2-\beta^2)\epsilon^{\mu\nu}\partial_\nu\sum_{\lambda=1}^{M}:\sin\{2(\alpha\tilde{\Sigma}(x)+\beta\tilde{\eta}(x)+\delta\tilde{\varphi}_\lambda(x))\}:|\Psi\rangle = 0\,,\end{aligned}$$

so that Maxwell's equations will be satisfied on $\mathscr{H}_{\rm phys}$ provided

$$\alpha = \beta = \delta/\sqrt{M} \tag{2.45}$$

This implies

$$\dim[\bar{\psi}\psi] = 1\,, \tag{2.46}$$

so that for these values of the parameters ψ_λ will satisfy the massive Dirac equation

$$(i\partial\!\!\!/ + eA\!\!\!/ - m)\psi_\lambda(x) = 0\,. \tag{2.47}$$

The gauge-invariant algebra of observables is generated by the set of operators $\{\hat{\psi}_\lambda(x), J_\mu(x), F_{\mu\nu}(x), D(x,y)\}$; due to the equations of motion (2.37), $F_{\mu\nu}$ is formally the same to that of the massless theory:

$$F_{\mu\nu}(x) = \frac{1}{e}\sqrt{\frac{\pi}{M}}\,\epsilon_{\mu\nu}\Box(\tilde{\Sigma}(x) + \tilde{\eta}(x)) = -e\sqrt{\frac{M}{\pi}}\,\epsilon_{\mu\nu}\tilde{\Sigma}(x)\,. \tag{2.48}$$

$D(x,y)$ is the bilocal operator formally given by

$$D(x,y) \sim \sum_{\lambda=1}^{M}\psi_\lambda(x)\exp\left[ie\int_x^y A_\mu(z)\,dz^\mu\right]\psi^+_\lambda(y)\,. \tag{2.49}$$

In order to cast $D(x,y)$ into a form exhibiting explicitly the *spurionization* of free charge for an arbitrary path of integration in the line integral, we write the fermion field ψ_λ in terms of the operator $\hat{\psi}_\lambda$:

$$\psi_\lambda(x) = :\hat{\psi}_\lambda(x)\exp\left[i\sqrt{\frac{\pi}{M}}\int_{x^1}^{\infty} dy^1\,\partial_0\tilde{\eta}(x^0,y^1)\right]:\,. \tag{2.50}$$

The exponential can be shown to be independent of the choice of integration-path up to a c-number phase [13]. Ignoring this phase one therefore has

$$\psi_\lambda(x) = :\hat{\psi}_\lambda(x) \exp\left[i\sqrt{\frac{\pi}{M}} \int_x^\infty \epsilon_{\mu\nu}\partial^\nu \tilde{\eta}(z)\, dz^\mu \right]: \tag{2.51}$$

The line integral over the $\tilde{\eta}$ field in eq.(2.51) cancels the corresponding η contribution in the line integral over the gauge field, thus leading to

$$D(x, y) = \mathcal{N}(x-y) \sum_{\lambda=1}^{M} :\exp\left[i\sqrt{\frac{\pi}{M}} \left\{ \gamma_x^5 \tilde{\chi}_\lambda(x) - \int_x^y \epsilon_{\mu\nu}\, \partial^\nu \tilde{\chi}_\lambda(z)\, dz^\mu - \gamma_y^5 \tilde{\chi}_\lambda(y) \right\} \right] :\sigma_x \sigma_y^+ ,$$

where $\mathcal{N}(x-y)$ is the matrix defined in ref. [14], and $\tilde{\chi}_\lambda(x)$ is given by

$$\tilde{\chi}_\lambda(x) = \tilde{\Sigma}(x) + \sqrt{M}\, \tilde{\varphi}_\lambda(x) - \tilde{\varphi}(x) . \tag{2.52}$$

σ is a spurion operator of the form of (2.17b) which commutes with all the observables. Therefore, on the physical subspace (2.10), it is a constant unitary operator which now only carries the charge selection rule, since the chiral symmetry is explicitly broken by the mass term. In an irreducible representation we can therefore replace $\sigma_2^+ \sigma_1$ by $e^{i\theta}$ and $\sigma_1 \sigma_2$ by $e^{i\bar{\theta}}$. The ground state is of course degenerate with respect to the angle θ which is associated with the spurionization of the free fermionic charge; it is non-degenerate with respect to $\bar{\theta}$, again reflecting the fact that the chiral symmetry is explicitly broken.

In contradistinction to the massless case, the analogue of the transformation (2.12)

$$\hat{\psi}_\lambda(x) = :\exp\left[i\sqrt{\frac{\pi}{M}} \int_{x^1}^\infty dy^1 \dot{\tilde{\eta}}(x^0, y^1) \right] \psi_\lambda(x): , \tag{2.53}$$

$$\hat{A}_\mu(x) = A_\mu(x) - \frac{1}{e}\sqrt{\frac{\pi}{M}}\, \partial_\mu \int_{x^1}^\infty dy^1 \dot{\tilde{\eta}}(x^0, y^1) ,$$

no longer leaves the Dirac equation invariant. This already follows from the fact that

$$[\hat{\psi}_\lambda(x), \mathcal{M}(y)] = -\frac{\mu}{T} \sum_{\lambda'=1}^{M} \left[:\cos\left(2\sqrt{\frac{\pi}{M}} \tilde{\chi}_{\lambda'}(y) + \theta + \Omega \right): - :\cos\left(2\sqrt{\frac{\pi}{M}} \tilde{\chi}_{\lambda'}(y) + \theta \right): \right] \hat{\psi}_\lambda(x) , \tag{2.54a}$$

where χ_λ has been defined in (2.52) and

$$\Omega = -\frac{2\pi}{M}\theta(y^1 - x^1)\,. \tag{2.54b}$$

For $M = 1$ the commutator vanishes identically. However, for $M > 1$, eq. (2.54) shows that $\hat{\psi}_\lambda(x)$ is in fact non-local with respect to the mass operator. Thus the transformation (2.54) can no longer be regarded as a gauge transformation in the usual sense, despite the fact that it does leave Maxwell's equations invariant.

The operator (2.53) still is of the form (2.16); it is invariant under c-number gauge transformations and therefore a candidate for creating physical states. In the massless case it does indeed create states of finite energy. However, in the massive case this is no longer true if $M > 1$ since according to the commutation relations (2.54) $\hat{\psi}$ then no longer leaves the mass operator invariant. We discuss the significance of this in subsect. 2.4.

2.4. Screening and confinement

Because of the non-zero mass of the field strength tensor $F_{\mu\nu}$, $\mathcal{H}_{\rm phys}$ only contains electrically neutral states [1]. In the case of massless fermions, the (gauge-invariant) operator (2.16) creates such a (finite-energy) state:

$$[Q, \hat{\psi}_\lambda(x)] = 0\,.$$

$\hat{\psi}_\lambda(x)$, however, carries the flavour of the fundamental fermion:

$$[Q^{i_{\rm D}}, \hat{\psi}_{\rm f}(x)] = -\tfrac{1}{2}\lambda^{i}_{\rm ff'}\hat{\psi}_{\rm f'}(x)\,.$$

Thus for $m = 0$, $\mathcal{H}_{\rm phys}$ contains zero-charge states belonging to the fundamental representation of SU(M) flavour. This means that the U(1) charge of the original U(M) fermion multiplet is merely screened by the vacuum polarization induced by the electromagnetic interaction and indicates screening without confinement.

The picture is quite different if the fermions are massive. In this case the operator $\hat{\psi}_\lambda(x)$ no longer creates a state of finite energy for $M > 1$, since it does not commute with the mass operator (see eq. (2.54)). This is a consequence of the line integral present in $\mathcal{F}_\lambda(x)$, eq. (2.17a), which generates translations in φ_λ space which are not symmetries of the mass term (for $x \to \infty$)

$$\mathcal{M} = \frac{-\mu}{\pi}\sum_\lambda \cos\left(2\sqrt{\frac{\pi}{M}}\,\tilde{\Sigma}(x) + 2\sqrt{\pi}\left(\tilde{\varphi}_\lambda - \frac{1}{\sqrt{M}}\tilde{\varphi}\right) + \theta\right). \tag{2.55}$$

However, for the special value $\theta = \pi/M$, $\mathcal{H}_{\rm phys}$ still contains zero-charge states transforming according to the fundamental representation of SU(M); they are obtained by applying the operators $K\hat{\psi}^{+}_\lambda$ and $\hat{\psi}_\lambda K$ on the $\theta = \pi/M$ vacuum where

$$K = K_\Sigma \prod_{\lambda=1}^{M} K_{\varphi_\lambda}\,,$$

where K_Φ is the "kink" operator [15] with the property

$$K_\Phi \Phi(x) K_\Phi^{-1} = \begin{cases} \Phi(x)\,, & x \to -\infty\,, \\ -\Phi(x)\,, & x \to +\infty\,, \end{cases} \tag{2.56}$$

The state $\hat{\psi}_\lambda K|\theta = -\pi/M\rangle$ carries the quantum numbers of the "anti-particle" corresponding to $K\hat{\psi}_\lambda^+|\theta = \pi/M\rangle$; it can be thought of as a $(M-1)$ particle bound state. On the other hand, $K\hat{\psi}_\lambda$ and $\hat{\psi}_\lambda^+ K$ create states of finite energy for $\theta = -\pi/M$. Thus, except for some special θ-worlds, the massive model exhibits confinement of the "exotic" states belonging to the fundamental representation of SU(M). The existence of such exotic states for SU(2) has already been noted previously [3].

On the other hand, the usual non-exotic states are not confined: For generic θ-worlds the operators $\hat{\psi}_\lambda^+ \hat{\psi}_{\lambda'}$, $\prod_{i=1}^M \hat{\psi}_{\lambda_i}(x_i)$ and $\prod_{i=1}^M \hat{\psi}_{\lambda_i}^+(x_i)$ create finite-energy states corresponding to "quark-antiquark", M-quark and M-antiquark bound states. Hence, except for some special θ-worlds, the massive model describes what one would expect from a more realistic theory: confinement of states carrying non-zero "triality".

The absence of exotic states is a result of the fact that the mass term (2.55) in the bosonized version of the theory now represents an additional "effective potential" which, except for special values of θ, grows linearly with the separation of two "exotic" particles, thus binding them permanently into hadrons. As far as the quantum numbers are concerned, the spectrum of states is in agreement with a confining picture. For the special values of $\theta = \pm\pi/M$ this bond is broken, allowing for the liberation of these exotic states. We are then left again with the "screened-quark" picture of the massless case.

Note that the θ-values $\theta = \pm(\pi/M)n$, $n = 0, 1, \ldots$, are precisely those values for which the model becomes P and T invariant. The fastest way to see this is to perform a SU(M) diagonal chiral rotation which transforms away all the θ's in (2.55) except the one in the last term of the sum which has to be

$$\frac{\pi}{M} + \frac{M-1}{M}\pi = \pi\,.$$

The mass term after this transformation is invariant under $\tilde{\Sigma} \to -\tilde{\Sigma}$, $\tilde{\varphi}_\lambda \to -\tilde{\varphi}_\lambda$.

3. Abelian gauge theory on the torus of SU(N)

3.1. The massless model

As the next step in the program outlined in sect. 1 we consider the extension of massless QED_2 to an Abelian gauge theory with SU$(N)_D$ as colour symmetry group, where SU$(N)_D$ denotes the diagonal subgroup, the torus of SU(N)

generated by the $N-1$ mutually commuting generators $\{\lambda^{i_D}\}$ of SU(N) satisfying (2.13). The corresponding Lagrangian is

$$\mathscr{L} = -\tfrac{1}{4}F^{i_D}_{\mu\nu}F^{i_D\mu\nu} + \tfrac{1}{2}\bar{\psi}i\gamma\cdot\partial\psi + g\bar{\psi}\gamma^{\mu}\tfrac{1}{2}\lambda^{i_D}\psi A^{i_D}_{\mu}\,, \tag{3.1}$$

where

$$F^{i_D}_{\mu\nu} = \partial_\mu A^{i_D}_\nu - \partial_\nu A^{i_D}_\mu\,, \tag{3.2}$$

and summation over the $N-1$ values of i_D is understood. The Lagrangian (3.1) is invariant only under local gauge transformations generated by the diagonal subgroup SU$(N)_D$ of SU(N). The corresponding (classical) equations of motion are

$$i\gamma\partial\psi + g\gamma^{\mu}\tfrac{1}{2}\lambda^{i_D}A^{i_D}_{\mu}\psi = 0\,, \tag{3.3a}$$

$$\partial^{\mu}F^{i_D}_{\mu\nu} + gJ^{i_D}_{\nu} = 0\,, \tag{3.3b}$$

where

$$J^{i_D}_{\nu} = \sum_{a=1}^{N} \bar{\psi}^a\gamma_\nu\tfrac{1}{2}\lambda^{i_D}\psi^a\,, \tag{3.3c}$$

with the superscript a representing the colour index.

The operator solution in the Schwinger gauge of the corresponding quantum equations is again a straightforward generalization of those discussed by Lowenstein and Swieca [4] for QED$_2$:

$$\psi^a_\alpha(x) = :\exp\left[i\sqrt{\tfrac{1}{2}\pi}\,\gamma^5_{\alpha\alpha}\sum_{i_D}\lambda^{i_D}_{aa}\,(\tilde{\Sigma}^{i_D}(x)+\tilde{\eta}^{i_D}(x))\right]:\psi^a_0(x)_\alpha\,, \tag{3.4a}$$

$$A^{i_D}_{\mu}(x) = -\frac{\sqrt{2\pi}}{g}\,\epsilon_{\mu\nu}\partial^{\nu}(\tilde{\Sigma}^{i_D}(x)+\tilde{\eta}^{i_D}(x))\,, \tag{3.4b}$$

where $\{\tilde{\Sigma}^{i_D}\}$ are $N-1$ free canonical pseudoscalar fields of mass $g^2/2\pi$,

$$\left(\Box+\frac{g^2}{2\pi}\right)\tilde{\Sigma}^{i_D}(x) = 0\,. \tag{3.5}$$

$\{\tilde{\eta}^{i_D}\}$ are $N-1$ free massless fields quantized with indefinite metric,

$$\Box\tilde{\eta}^{i_D}(x) = 0 \tag{3.6}$$

and $\psi^a_0(x)_\alpha$ are free massless fermion fields.

It is again convenient to write $\psi^a_0(x)$ in the bosonized form

$$\psi^a_0(x)_\alpha = \left(\frac{\mu}{2\pi}\right)^{1/2}\exp\left[-\tfrac{1}{4}i\pi\gamma^5_{\alpha\alpha}\right]:\exp\left[i\sqrt{\pi}\left(\gamma^5_{\alpha\alpha}\tilde{\varphi}^a + \int_{x^1}^{\infty}\mathrm{d}y^1\partial_0\tilde{\varphi}^a\right)\right]:\,, \tag{3.7}$$

where $\tilde{\varphi}^a$ are N free zero-mass fields related to the $N-1$ potentials $\tilde{\phi}^{i_D}$ of the free fermionic currents

$$j_\mu^{i_D}(x) = :\bar{\psi}_0(x)\,\tfrac{1}{2}\lambda^{i_D}\gamma_\mu\psi_0(x): = -\frac{1}{\sqrt{2\pi}}\epsilon_{\mu\nu}\partial^\nu\tilde{\phi}^{i_D}(x)\,, \tag{3.8}$$

by

$$\tilde{\phi}^{i_D} = \sqrt{\tfrac{1}{2}}\sum_a \lambda^{i_D}_{aa}\tilde{\varphi}^a \tag{3.9}$$

It will turn out convenient to introduce also the potential $\tilde{\phi}$ of the isocalar current

$$j_\mu = -\sqrt{\frac{N}{\pi}}\epsilon_{\mu\nu}\partial^\nu\tilde{\phi}\,; \tag{3.10}$$

$\tilde{\varphi}^a$, $\tilde{\phi}^i$ and $\tilde{\phi}$ are related by

$$\tilde{\varphi}^a = \frac{1}{\sqrt{N}}\tilde{\phi} + \sqrt{\tfrac{1}{2}}\sum_a \lambda^{i_D}_{aa}\tilde{\phi}^{i_D}\,, \tag{3.11}$$

and satisfy the equal-time commutation relations

$$\begin{aligned}
&[\tilde{\phi}(x), \tilde{\phi}^{i_D}(y)]_{ET} = 0\,,\\
&[\tilde{\phi}^i(x), \partial_0\tilde{\phi}^j(y)]_{ET} = i\delta^{ij}\delta(x^1 - y^1)\,,\\
&[\tilde{\varphi}(x), \partial_0\tilde{\varphi}^b(y)]_{ET} = i\delta^{ab}\delta(x^1 - y^1)\,.
\end{aligned} \tag{3.12}$$

The currents $J_\mu^{i_D}(x)$ appearing in the generalized Maxwell equations (3.3b), as obtained by the usual gauge-*invariant* limiting procedure, are calculated to be

$$J_\mu^{i_D}(x) = -\sqrt{\frac{1}{2\pi}}\epsilon_{\mu\nu}\partial^\nu\tilde{\Sigma}^{i_D}(x) + L_\mu^{i_D}(x)\,, \tag{3.13}$$

where the longitudinal part of the current is given by

$$L_\mu^{i_D}(x)_L = -\sqrt{\frac{1}{2\pi}}\partial_\mu(\phi^{i_D}(x) + \eta^{i_D}(x))\,, \tag{3.14}$$

with

$$\epsilon_{\mu\nu}\partial^\nu\tilde{\eta}^{i_D} = \partial_\mu\eta^{i_D}\,, \qquad \epsilon_{\mu\nu}\partial^\nu\tilde{\phi}^{i_D} = \partial_\mu\phi^{i_D}\,. \tag{3.15}$$

Hence the equations of motion are satisfied only on the physical subspace defined by

$$\langle\Phi|L_\mu^{i_D}(x)|\Psi\rangle = 0\,, \qquad |\Psi\rangle, |\Phi\rangle \in \mathscr{H}_{\text{phys}}\,. \tag{3.16}$$

Performing, as in the case of QED$_2$, the operator gauge transformation

$$\psi^a_\alpha(x) \to \exp\left[i\sqrt{\tfrac{1}{2}\pi}\sum_a \lambda^{i_D}_{aa}\eta^{i_D}(x)\right]\psi^a_\alpha(x)\,,$$

$$A^{i_D}_\mu(x) \to A^{i_D}_\mu(x) + \frac{\sqrt{2\pi}}{g}\partial_\mu \eta^{i_D}(x)\,,$$

one arrives at the operator solution in the "physical" gauge which is conveniently written in the form

$$\hat{\psi}^a(x) = \left(\frac{\mu}{2\pi}\right)^{1/2} \exp\left[-\tfrac{1}{4}i\pi\gamma^5\right]: \exp\left[i\sqrt{\tfrac{1}{2}\pi}\sum \lambda^{i_D}_{aa}\,\gamma^5\tilde{\Sigma}^{i_D}(x)\right]: S(x)\sigma^a(x)\,, \quad (3.17\text{a})$$

$$\hat{A}^{i_D}_\mu(x) = -\frac{\sqrt{2\pi}}{g}\epsilon_{\mu\nu}\partial^\nu\,\tilde{\Sigma}^{i_D}(x)\,, \quad (3.17\text{b})$$

where

$$S(x) = :\exp\left[i\sqrt{\frac{\pi}{N}}\left(\gamma^5\tilde{\phi}(x) + \int_{x^1}^{\infty} dy^1\partial_0\tilde{\phi}\right)\right]: \quad (3.17\text{c})$$

$$\sigma^a(x) = :\exp\left\{i\sqrt{\tfrac{1}{2}\pi}\sum_{i_D}\lambda^{i_D}_{aa}\left[\gamma^5(\tilde{\phi}^{i_D}(x) + \tilde{\eta}^{i_D}(x)) + \int_{x^1}^{\infty} dy^1\partial_0\left(\tilde{\phi}^{i_D} + \tilde{\eta}^{i_D}\right)\right]\right\}: \quad (3.17\text{d})$$

One novel feature of this Abelian generalization of QED$_2$ is the appearance of the "soliton-like" operator $S(x)$ which, as we shall see carries the charge selection rule associated with the fundamental fermion. $\sigma^a(x)$ is a constant operator on $\mathcal{H}_{\text{phys}}$ and generates again the different vacuum sectors of the theory.

3.2. Properties of the massless model

We now briefly discuss some of the interesting features of the solution.

3.2.1. Vacuum structure and clustering. The physical Hilbert space contains all finite-energy states which satisfy condition (3.16) and are invariant under local gauge transformations belonging to SU(N)$_D$. It is obtained by applying polynomials of

$$F_{\mu\nu}\,,\quad J^{i_D}_\mu\,,\quad L^{i_D}_\mu\,,\quad \hat{\psi}^a_\alpha(x)\,,\quad S\quad \text{and}\quad \sigma^a_\alpha\,,$$

as well as dipole-like operators on the Fock-vacuum. σ^a_α are constant operators on $\mathcal{H}_{\text{phys}}$ and carry the colour part of the fermionic solution rule. By applying σ^a_α on the Fock-vacuum, we generate an infinite set of (degenerate) ground states:

$$|(n_1^a), (n_2^b)\rangle = \prod_{a,b} (\sigma_1^a)^{n_1^a}(\sigma_2^b)^{n_2^b}|0\rangle\,. \tag{3.18}$$

Negative values of n_α^a correspond to the application of σ_α^{a+} on $|0\rangle$.

It follows from the commutation relations

$$[\tilde{Q}_5^{i_D}, \sigma_\alpha^a(x)] = -\gamma_{\alpha\alpha}^5\,\tfrac{1}{2}\lambda_{aa}^{i_D}\,\sigma_\alpha^a\,, \tag{3.19}$$

with

$$\tilde{Q}_5^{i_D} = \int dx^1\,\bar{\psi}_0\gamma^5\gamma^0\tfrac{1}{2}\lambda^{i_D}\psi_0\,, \tag{3.20}$$

that the ground states (3.18) are eigenstates of the charge operators $\tilde{Q}_5^{i_D}$ associated with the gauge-variant (free) axial currents $j_{5\mu}^{i_D}$. The coherent superposition of these states,

$$|\boldsymbol{\theta}_1, \boldsymbol{\theta}_2\rangle = \sum_{-\infty}^{\infty} e^{-i\boldsymbol{n}_1\cdot\boldsymbol{\theta}_1 - i\boldsymbol{n}_2\cdot\boldsymbol{\theta}_2}|\boldsymbol{n}_1, \boldsymbol{n}_2\rangle\,, \tag{3.21}$$

provides, as in the analogue case of QED_2 [4], a diagonal basis for σ_α^a and for the observables. We shall denote this vacuum globally by $|\theta\rangle$.

Whereas in the case of SU(M) flavour, the U(1) chirality and bare charge of the fermion was "dumped" into the vacuum, this occurs now for $SU(N)_D$ chirality (eq. (3.19)) and colour:

$$[Q^{i_D}, \sigma_\alpha^a] = -\tfrac{1}{2}\lambda_{aa}^{i_D}\sigma_\alpha^a\,. \tag{3.22}$$

On the other hand, the U(1) chiral and electric charge of the fundamental fermion is now carried by the colour-singlet "soliton"-operator

$$\begin{aligned}[][Q, S(x)] &= -S(x)\,,\\ [Q_5, S_\alpha(x)] &= -\gamma_{\alpha\alpha}^5 S_\alpha(x)\,.\end{aligned} \tag{3.23}$$

It is now the selection-rules carried by $S(x)$ which lead to a vanishing vacuum expectation value of $\hat{\psi}$, $\langle\theta|\hat{\psi}_\alpha^a|\theta\rangle = 0$ and prevents the gauge-invariant fermion two-point function from violating the cluster decomposition:

$$\langle\theta|T\hat{\psi}^a(x)\hat{\bar{\psi}}^b(0)|\theta\rangle \underset{|x|\to\infty}{\longrightarrow} \frac{\mu}{2\pi}\begin{pmatrix} 0 & e^{+i(\theta_1^a-\theta_1^b)}[i\mu(x^0+x^1)]^{-1/N} \\ e^{+i(\theta_2^a-\theta_2^b)}[i\mu(x^0-x^1)]^{-1/N} & 0\end{pmatrix}. \tag{3.24}$$

Violation of clustering can now only occur for operators carrying zero U(1) chirality and charge.

The state $\hat{\psi}^a(x)|0\rangle$ carries one unit of electric charge. As we shall show, this charge gets dumped into the vacuum if an electromagnetic interaction is introduced: the soliton operator which prevented a total breakdown of the cluster property in (3.24) is thereby turned into a "spurion".

3.2.2. Gauge transformation. The operator

$$T[\Lambda]=\exp\left[\frac{i}{\sqrt{2\pi}}Q[\Lambda]\right],$$

$$Q[\Lambda]=\int_{y^0=x^0} dy^1\{(\tilde{\phi}^{i_D}(y)+\tilde{\eta}^{i_D}(y))\partial_1\Lambda^{i_D}(y)-(\phi^{i_D}(y)+\eta^{i_D}(y))\partial_0\Lambda^{i_D}(y)\},\tag{3.25}$$

induces the c-number gauge transformation

$$T[\Lambda]\psi^a(x)T^{-1}[\Lambda]=\exp[i\tfrac{1}{2}\lambda^{i_D}_{aa}\Lambda^{i_D}(x)]\psi^a(x),$$

$$T[\Lambda]A^{i_D}_\mu(x)T^{-1}[\Lambda]=A^{i_D}_\mu(x)+\frac{1}{g}\partial_\mu\Lambda^{i_D}(x),\tag{3.26}$$

where ψ^a and $A^{i_D}_\mu$ are the fermion field and vector potential in the Schwinger-gauge . We observe that $\hat{\psi}^a_\alpha$ and $\hat{A}^{i_D}_\mu$ in eqs. (3.17) are left invariant by this c-number transformation and thus represent observables of the theory. As in the case of the Schwinger-model, the operators σ^a_α can be identified on $\mathscr{H}_{\text{phys}}$ with $T[\Lambda]$ for particular choices of Λ [10].

3.3.3. SU(N) chiral transformations. The gauge-invariant axial-current $\boldsymbol{J}_5^\mu$ is given by (see eqs. (3.46), (3.13))

$$\boldsymbol{J}_5^\mu=\epsilon^{\mu\nu}\boldsymbol{J}_\nu=-\frac{1}{\sqrt{2\pi}}\epsilon^{\mu\nu}\partial_\nu\boldsymbol{\phi}-\frac{g}{2\sqrt{\pi}}\epsilon^{\mu\nu}\boldsymbol{A}_\nu,$$

where the vector symbol stands for the indices (i_D). Because of the anomaly

$$\partial_\mu\boldsymbol{J}_5^\mu=-\frac{g}{4\pi}\epsilon^{\mu\nu}\boldsymbol{F}_{\mu\nu},\tag{3.27}$$

the associated axial charge is not conserved and, hence does not generate chiral transformations. Such transformations are generated instead by the axial charge $\tilde{Q}_5$, eq. (3.20):

$$\psi^a_\alpha(x)\to\exp[-\tfrac{1}{2}i\gamma^5_{\alpha\alpha}\boldsymbol{\lambda}_{aa}\cdot\boldsymbol{\vartheta}]\psi^a_\alpha(x).\tag{3.28}$$

The ground state transforms correspondingly as

$$|(\theta^a_1),(\theta^b_2)\rangle\to|(\theta^a_1+\tfrac{1}{2}\boldsymbol{\lambda}_{aa}\cdot\boldsymbol{\vartheta}),(\theta^b_2-\tfrac{1}{2}\boldsymbol{\lambda}_{bb}\cdot\boldsymbol{\vartheta})\rangle.\tag{3.29}$$

3.3. Extension to $SU(N)_D\times U(1)$

In the following we shall identify the "soliton" quantum number with the charge of the fermion by adding an electromagnetic interaction to (3.1):

$$\mathscr{L}'=\mathscr{L}-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}+e\bar{\psi}\gamma^\mu\psi A_\mu.$$

We now obtain for the solution in the "physical gauge"

$$\hat{\psi}'(x)^a = \left(\frac{\mu}{2\pi}\right)^{1/2} :$$

$$\times \exp[-i\tfrac{1}{4}\pi\gamma^5] : \exp[i\tfrac{1}{2}\pi\gamma^5\lambda^{i_D}_{\alpha a}\tilde{\Sigma}^{i_D}] : :\exp\left[i\sqrt{\frac{\pi}{N}}\gamma^5\tilde{\Sigma}\right] : \sigma(x)\sigma^a(x) : ,$$

$$\hat{A}_\mu = -\frac{1}{e}\sqrt{\frac{\pi}{N}}\epsilon_{\mu\nu}\partial^\nu\tilde{\Sigma} ,$$

$$\left(\Box + \frac{Ne^2}{\pi}\right)\tilde{\Sigma} = 0 , \tag{3.30}$$

with $\hat{A}^{i_D}_\mu$ and σ^a_α still given by eqs. (3.17b, c) and

$$\sigma(x) = \exp\left[i\sqrt{\frac{\pi}{N}}[\gamma^5(\tilde{\phi} + \tilde{\eta}) + \int_{x^1}^{\infty} dy^1\, \partial_0(\tilde{\phi} + \tilde{\eta})]\right] . \tag{3.31}$$

The choice of constants ensures again a canonical short-distance behaviour. The corresponding electromagnetic current is now

$$J^\mu = -\sqrt{\frac{N}{\pi}}\partial^\mu(\phi + \eta) - \sqrt{\frac{N}{\pi}}\epsilon^{\mu\nu}\partial_\nu\tilde{\Sigma} .$$

Condition (3.10) now has to be supplemented by a corresponding condition for $L^\mu = -\sqrt{N/\pi}\,\partial^\mu(\phi + \eta)$. Taking account of this additional constraint one sees that the "soliton" operator $S(x)$ of the original solution (3.17) has spurionized, i.e., has dumped its quantum number into the vacuum, which now carries in addition a charge and $U(1)$ chiral solution rule. $\mathscr{H}_{\text{phys}}$ now no longer contains states of non-zero charge, a property that can again be traced to the Coulomb interaction in the Hamiltonian. The quantum number carried by the soliton operator has thus been screened by the electromagnetic interaction, which allows one to identify this quantum number with the basic fermionic charge. There will now be an additional angle θ characterizing the θ-vacuum, which we now globally denote by $|\boldsymbol{\theta}, \theta\rangle$. Unlike in the case of $SU(N)_D$, $\hat{\psi}'$ now has a non-vanishing expectation value with respect to the new vacuum:

$$\langle\boldsymbol{\theta}, \theta|\hat{\psi}'(x)|\boldsymbol{\theta}, \theta\rangle \neq 0 . \tag{3.32}$$

$\hat{\psi}'(x)$ may thus be viewed as a disorder variable, with $\hat{\psi}'$ and $|\boldsymbol{\theta}, \theta\rangle$ playing a role analogous to that of the Cooper pair and superconducting ground state in the BCS theory of superconductivity. The non-vanishing expectation value (3.32) now implies a violation of clustering.

3.4. Adding a fermion mass

Using again methods analogous to those of sect. 2 one arrives at a formal solution for the massive theory in the Schwinger gauge:

$$\psi^a(x)=\left(\frac{\mu}{2\pi}\right)^{1/2}\exp\left[-\tfrac{1}{4}i\pi\gamma^5\right]:\exp\left[i\sqrt{\tfrac{1}{2}\pi}\gamma^5\sum_a\lambda^{i_D}_{aa}(\tilde{\Sigma}^{i_D}+\tilde{\eta}^{i_D})+i\sqrt{\pi}\left(\gamma^5\tilde{\varphi}^a+\int_{x^1}^{\infty}dz^1\partial_0\tilde{\varphi}^a\right)\right]:,$$

$$A^{i_D}_\mu=-\frac{\sqrt{2\pi}}{g}\epsilon_{\mu\nu}\partial^\nu(\tilde{\Sigma}^{i_D}+\tilde{\eta}^{i_D}),\tag{3.33}$$

which is formally the same that given by eqs. (3.4) and (3.7), except that $\tilde{\Sigma}^{i_D}$, $\tilde{\varphi}^a$ and $\tilde{\eta}^{i_D}$ are no longer free fields, but satisfy the coupled set of equations

$$\left(\Box+\frac{g^2}{2\pi}\right)\tilde{\Sigma}^{i_D}=-\mu m\sqrt{\frac{2}{\pi}}\sum_a\lambda^{i_D}_{aa}\sin(2\sqrt{\pi}\tilde{\Phi}^a),\tag{3.34a}$$

$$\Box\tilde{\varphi}^a=-\frac{2\mu m}{\sqrt{\pi}}\sin(2\sqrt{\pi}\tilde{\Phi}^a),\tag{3.34b}$$

$$\Box(\tilde{\eta}^{i_D}+\tilde{\phi}^{i_D})=0,\tag{3.34c}$$

with $\tilde{\phi}^{i_D}$ given by eq. (3.9a) and

$$\tilde{\Phi}^a=\tilde{\varphi}^a+\sqrt{\tfrac{1}{2}}\sum_{i_D}\lambda^{i}_{aa}(\tilde{\Sigma}^{i_D}+\tilde{\eta}^{i_D}).\tag{3.35}$$

As in the case of flavour one finds that the mass operator

$$\mathcal{M}=N[\bar{\psi}\psi]=-\frac{\mu}{\pi}\sum_a:\cos(2\sqrt{\pi}\tilde{\Phi}^a):\tag{3.36}$$

has a canonical scale dimension, and as a result $\psi^a(x)$ satisfies the equation of motion

$$(i\partial\!\!\!/+g\tfrac{1}{2}\lambda^{i_D}A\!\!\!/^{i_D}-m)\Psi=0,$$

where normal ordering is implied.

The $N-1$ mutually commuting "diagonal" currents are calculated to be

$$J^{i_D}_\mu=-\frac{1}{\sqrt{2\pi}}\epsilon_{\mu\nu}\partial^\nu(\tilde{\Sigma}^{i_D}+\tilde{\eta}^{i_D}+\tilde{\phi}^{i_D}).$$

Maxwell's equations are again satisfied on $\mathcal{H}_{\rm phys}$ defined by condition (3.16) where $L^{i_D}_\mu$ is a *free* field as seen from eq. (3.34c).

The operator σ_α^a defined by eq. (3.17d) still commutes with the Hamiltonian. However, in the massive case the Hamiltonian itself now involves σ_α^a (through the mass term) so that the original vacuum degeneracy has been removed by the mass perturbation which now induces tunnelling transitions between the original imperturbed n-vacua.

The ground states are degenerate with respect to the $N-1$ angles $\tilde{\theta}^a$ associated with the spurionization of the free fermion colour, but depend explicitly on $N-1$ angles θ^a as a result of the explicit breaking of SU(N) chiral invariance by the mass term.

3.5. Screening and confinement

Whereas in the case of QED_2 with SU(M) flavour the gauge field coupled to the charge of the fermions, the gauge fields in the case of $SU(N)_D$ couple to the colour. Hence, the role of the fermionic charge in SU(M) flavour is now taken up by the colour. $\mathscr{H}_{phys}$ only contains colour-singlet states, since operators carrying colour (analogue of Coulomb gauge operators) create states of infinite energy. The electric charge of the fermions, however, remains unscreened; these physical states thus belong to the (one-dimensional) irreducible representations of $U(1)\times SU(N)$.

In the case of zero-mass fermions the operator $\hat{\psi}^a$ in (3.17a) provides an example of an operator creating a physical colour-singlet state carrying one unit of charge:

$$[Q^i, \hat{\psi}^a(x)] = 0\,, \qquad [Q, \hat{\psi}^a(x)] = -\hat{\psi}^a(x)\,.$$

This indicates screening of "quark-colour" rather than confinement. However, when introducing a fermion mass, $\hat{\psi}^a(x)$ no longer creates a finite-energy state, since the "soliton" operator (3.17c) generates the translation $\tilde{\phi} \to \tilde{\phi} + \sqrt{\pi/N}$ which does not leave the mass term

$$\mathscr{M} = -\frac{\mu}{\pi}\sum_a :\cos\left(2\sqrt{\frac{\pi}{N}}\,\tilde{\phi}(x) + \sqrt{2\pi}\sum_{i_D}\lambda^{i_D}_{aa}\tilde{\Sigma}^{i_D}(x) + \theta^a\right):$$

invariant; in more physical terms: even though no Coulomb forces are present, the pair of charged particles created by the operators $\hat{\psi}^a$ and $\hat{\psi}^{a\dagger}$ see a linearly rising (Coulomb-like) potential between them. Since the electric charge cannot be screened by the interaction with the gluons, this means that these singly charged states have become confined. Nevertheless there again exist special values of θ for which $\mathscr{H}_{phys}$ contains (colour screened quark) states carrying one unit of charge: remember that the θ^a angles are not all independent ($\sum_a \theta^a = 0$), one finds that the operators $K\hat{\psi}^{a\dagger}$ and $\hat{\psi}^a K$ create charged finite-energy states if $N-1$ of the $N\theta^a$'s have the value $-\pi/N$, where

$$K = \prod_{i=1}^{N-1} K_{\Sigma^i} K_{\phi}\,,$$

where K_Φ is the "kink" operator (2.56).

In addition to these "exotic" states, which exist only for particular θ-worlds, $\mathscr{H}_{\text{phys}}$ contains the usual non-exotic states corresponding to "quark-antiquark", N-quark and N-antiquark bound states. They are obtained by applying the operators $\hat{\psi}^{a}(x)^{+}\hat{\psi}^{b}(x)$ and $\prod_{i=1}^{N}\hat{\psi}^{a_i}(x)$ on the ground state and carry a zero and N-tuple fermionic charge, respectively.

The presence or absence of exotic states above can intuitively be understood in a way similar to the one discussed in subsect. 2.5.

4. Extension to $SU(M)_F\times SU(N)_{D,G}$

In the following we summarize the results one obtains by combining those of sects. 2 and 3. We shall immediately turn to the massive case, since the case of zero-mass fermions is just a particular case of the general one.

4.1. The solution

Except for the addition of a mass term $m\bar{\psi}\psi$, the Lagrangian corresponding to a $SU(M)_F\times SU(N)_{D,G}$ symmetry is again of the form (3.1) where $\psi^{a}_{\lambda}(x)_\alpha$ now carries an additional flavour degree of freedom λ. The Schwinger gauge solution to the corresponding equations of motion is obtained as a straightforward generalization of the solutions constructed in the preceding sections:

$$\psi^{a}_{\lambda}(x)=\left(\frac{\mu}{2\pi}\right)^{1/2}\exp\left[-\tfrac{1}{4}i\pi\gamma^5\right]\!:\exp\left[i\sqrt{\frac{\pi}{2M}}\gamma^5\sum_{i_D}\lambda^{i_D}_{aa}\left(\tilde{\Sigma}^{i_D}(x)+\tilde{\eta}^{i_D}(x)\right)\right]$$
$$\times\exp\left[i\sqrt{\pi}\left\{\gamma^5\tilde{\varphi}^{a}_{\lambda}(x)+\int_{x^1}^{\infty}\mathrm{d}y^1\,\partial_0\tilde{\varphi}^{a}_{\lambda}(y^1,x^0)\right\}\right]:,\tag{4.1a}$$

$$A^{i_D}_{\mu}(x)=-\frac{1}{g}\sqrt{\frac{2\pi}{M}}\left\{\epsilon_{\mu\nu}\partial^{\nu}\tilde{\Sigma}^{i_D}(x)+\partial_{\mu}\tilde{\eta}^{i_D}(x)\right\},\tag{4.1b}$$

where the boson fields now satisfy the equations

$$\left(\Box+\frac{Mg^2}{2\pi}\right)\tilde{\Sigma}^{i_D}(x)+m\mu\sqrt{\frac{2}{\pi M}}\sum_{\lambda,a}\lambda^{i_D}_{aa}:\sin\left(2\sqrt{\pi}\tilde{\Phi}^{a}_{\lambda}\right):=0,$$
$$\Box\tilde{\eta}^{i_D}(x)-m\mu\sqrt{\frac{2}{\pi M}}\sum_{\lambda,a}\lambda^{i_D}_{aa}:\sin\left(2\sqrt{\pi}\tilde{\Phi}^{a}_{\lambda}\right):=0,$$
$$\Box\tilde{\varphi}^{a}_{\lambda}(x)+\frac{2m\mu}{\sqrt{\pi}}:\sin\left(2\sqrt{\pi}\tilde{\Phi}^{a}_{\lambda}\right):=0,\tag{4.2}$$

where

$$\tilde{\Phi}^{a}_{\lambda}(x)=\tilde{\varphi}^{a}_{\lambda}(x)+\sqrt{\frac{1}{2M}}\sum_{i_D}\lambda^{i_D}_{aa}\left(\tilde{\Sigma}^{i_D}(x)+\tilde{\eta}^{i_D}(x)\right).$$

From (4.1) one finds for the gauge-invariant current

$$J_\mu^{i_D}(x) = -\sqrt{\frac{M}{2\pi}}\,\epsilon_{\mu\nu}\partial^\nu \tilde{\Sigma}^{i_D}(x) + L_\mu^{i_D}(x)\,,$$

where

$$L_\mu^{i_D}(x) = -\sqrt{\frac{M}{2\pi}}\,\epsilon_{\mu\nu}\partial^\nu(\tilde{\phi}^{i_D}(x) + \tilde{\eta}^{i_D}(x)\,,$$

with

$$\tilde{\phi}^{i_D} = \sqrt{\tfrac{1}{2}}\sum_a \lambda_{aa}^{i_D}\tilde{\varphi}^a\,, \qquad \tilde{\varphi}^a = \frac{1}{\sqrt{M}}\sum_\lambda \tilde{\varphi}_\lambda^a\,.$$

Thus Maxwell's equations will be satisfied on $\mathscr{H}_{\text{phys}}$ defined by (3.16). Observe that on account of equations of motion (4.2), $L_\mu^{i_D}$ is again a purely longitudinal free field creating states of zero norm.

Introducing further the canonically quantized fields $\tilde{\phi}_\lambda$ and $\tilde{\phi}$ *via* the relations

$$\tilde{\varphi}_\lambda^a(x) = \frac{1}{\sqrt{M}}\tilde{\phi}_\lambda(x) + \sqrt{\tfrac{1}{2}}\sum_{i_D} \lambda_{aa}^{i_D}\tilde{\phi}_\lambda^{i_D}(x),$$

$$\tilde{\phi}(x) = \frac{1}{\sqrt{M}}\sum_{\lambda=1}^{M} \phi_\lambda(x)\,,$$

we may write the gauge-invariant operator $\hat{\psi}_\lambda^a$ corresponding to (4.1a) in the factorized form

$$\hat{\psi}_\lambda^a(x) = \exp[-\tfrac{1}{4}i\pi\gamma^5]\!:\!\exp\left[i\sqrt{\frac{\pi}{2M}}\gamma^5\sum_{i_D}\lambda_{aa}^{i_D}\tilde{\Sigma}^{i_D}(x)\right]\!:\!\mathscr{F}_\lambda^a(x)S(x)\sigma^a,$$

where

$$\mathscr{F}_\lambda^a(x) = :\!\exp\left[i\sqrt{\pi}\gamma^5\left(\tilde{\varphi}_\lambda^a(x) - \frac{1}{\sqrt{M}}\tilde{\varphi}^a(x)\right) + i\sqrt{\pi}\int_{x^1}^{\infty} dy^1\partial_0\left(\tilde{\varphi}_\lambda^a - \frac{1}{\sqrt{M}}\tilde{\varphi}^a\right)\right]\!:\,,$$

$$S(x) = :\exp\left[i\sqrt{\frac{\pi}{MN}}(\gamma^5\tilde{\phi}(x) + \int_{x^1}^{\infty} dy^1\,\partial_0\tilde{\phi}(y^1, x^0))\right]\!:$$

$$\sigma^a = \exp\left[i\sqrt{\frac{\pi}{2M}}\sum_{i_D}\lambda_{aa}^{i_D}\{\gamma^5(\tilde{\phi}^{i_D}(x) + \tilde{\eta}^{i_D}(x)) + \int_{x^1}^{\infty} dy^1\partial_0(\tilde{\phi}^{i_D} + \tilde{\eta}^{i_D})\}\right].$$

The vacuum again carries the quantum numbers of the "spurion" operator σ_α^a, that is colour and $SU(N)_C$ chirality. The "soliton" operator carries the fundamental fermionic charge and U(1) chirality; it turns into a spurion when an electromagnetic interaction is introduced:

$$\hat{\psi}'^{a}_{\lambda}(x) = \exp\left[-\tfrac{1}{4} i\pi\gamma^5\right] : \exp\left[i\sqrt{\frac{\pi}{2M}}\gamma^5 \sum_{i_D} \lambda^{i_D}_{aa} \tilde{\Sigma}^{i_D}(x)\right] : \mathscr{F}^{a}_{\lambda}(x)\sigma^{a}\sigma .$$

The θ-vacuum now carries also the quantum numbers of the soliton.

The operator $\mathscr{F}^{c}_{\lambda}(x)$ satisfies the commutation relations

$$[Q^{i_D}, \mathscr{F}^{a}_{\lambda}(x)] = 0 ,$$

$$[\bar{Q}^{i_D}, \mathscr{F}^{c}_{\lambda}(x)] = -\tfrac{1}{2}\lambda^{i_D}_{\lambda\lambda}\mathscr{F}^{c}_{\lambda} ,$$

where Q^{i_D}, $\bar{Q}^{i_D}$ are the "diagonal" generators of SU(N) colour and SU(M) flavour, respectively. Hence, the operator $\mathscr{F}^{a}_{\lambda}$ is complementary to σ^{a}_{α} in the sense that it carries flavour, but no colour.

Hence, $\mathscr{F}^{a}_{\lambda}(x)$ and $S(x)$ still carry the same selection rules as in the pure flavour and pure colour models discussed previously. They now jointly control the cluster properties of the correlation functions.

4.2. Screening and confinement

In the absence of an electromagnetic interaction, $\mathscr{H}_{\text{phys}}$ only contains electrically charged, colour-singlet states. For a zero fermion mass the operator $\hat{\psi}^{a}_{\lambda}(x)$ creates such a colour-singlet state carrying one unit of charge: it belongs to the fundamental representation of U(1)$\times$SU(N) flavour. When the fermions are massive, this operator no longer creates a finite-energy state. Nevertheless, finite-energy colour-screened "quark" states can again be constructed for particular choices of θ-worlds. Such states are

$$\begin{aligned} &K\hat{\psi}^{a\dagger}_{\lambda}(x)|\{\theta^{c}\}\rangle \quad \text{for} \quad \theta^{b} = -\frac{\pi}{MN}, \quad b \neq a , \\ &\hat{\psi}^{a}_{\lambda}(x)K|\{\theta^{c}\}\rangle \quad \text{for} \quad \theta^{b} = +\frac{\pi}{MN}, \; b \neq a , \end{aligned} \tag{4.3}$$

as well as their respective "antiparticle" counterparts, where it is to be kept in mind that $\sum^{N}\theta^{c} = 0$ and

$$K = \prod_{i=1}^{N-1} K_{\tilde{\Sigma}^{i}} \prod_{c}^{N}\prod_{\lambda}^{M} K_{\varphi^{c}_{\lambda}} K\phi .$$

The operators $\hat{\psi}^{a}_{\lambda}$, although colour singlets, still carry a selection rule associated with the "colour index" a: this is the consequence of the existence of a conserved (topological) current, the free fermion current $j^{a}_{\lambda}(x)_{\mu}$, which, however, does not generate symmetry transformations of the Lagrangian. As eq. (4.3) shows, the special θ-worlds (4.3) are very selective: in a world characterized by $\theta^{a} =$

$-(\pi/MN)(N-1)$ and $\theta^b = +\pi/MN$, all $b \neq a$, only quark-like states with topological colour index "a" exist.

The above selective character of the θ-vacua persists if an additional electromagnetic interaction is introduced. The θ-vacuum is now characterized by an additional angle θ. This additional degree of freedom allows again for special θ-worlds containing exotic states. They correspond to

$$\begin{aligned} &K\hat{\psi}'^a_\lambda(x)^\dagger|\{\theta^c\},\theta\rangle\,, \qquad \theta=\frac{\pi}{MN}\,, \qquad \theta^b=-\frac{\pi}{MN}\,, \qquad b\neq a\,,\\ &\hat{\psi}'^a_\lambda(x)^\dagger K|\{\theta^c\},\theta\rangle\,, \qquad \theta=-\frac{\pi}{MN}\,, \qquad \theta^b=+\frac{\pi}{MN}\,, \qquad b\neq a\,, \end{aligned} \tag{4.4}$$

as well as their respective "antiparticle" counterparts, where again $\sum^N\theta^c=0$ and

$$K=K_{\hat{\Sigma}}\prod_{c=1}^{N-1}K_{\hat{\Sigma}^i}\prod_{\lambda=1}^{M}\prod_{c=1}^{N}K_{\hat{\varphi}^c_\lambda}\,.$$

In addition to the above exotic states there exist, of course, the usual non-exotic "bound" states for arbitrary values of θ.

Similarly to the consideration at the end of subsect. 2.4, the "exotic" values for θ can be shown to be precisely those at which the model becomes P and T invariant.

5. Connection with QCD_2

In QCD with SU(N) as gauge group there always exists a gauge in which $F^i_{\mu\nu}(x)$ points in the direction of the torus of SU(N). This, however, is not true for the vector potential $A^i_\mu(x)$, unless the local gauge symmetry is to be broken down to the torus of SU(N). Since a *local* gauge symmetry cannot be broken (by definition!), our torus solution cannot be regarded as a solution of spontaneously broken QCD_2* . Nevertheless, we expect some features of our solution to be characteristic of QCD_2 as well. This concerns in particular the intrinsic Higgs mechanism exhibited by the Abelian model which prevents the existence of states carrying colour. As we now show, a similar phenomenon is expected to occur in QCD_2; it is intimately connected with the axial current anomaly.

We define the axial vector current by the following covariant limiting procedure:

$$J^i_{5\mu}(x)=\lim_{\epsilon\to 0}J^i_{5\mu}(x;\epsilon)\,,$$

* After finishing this work we received a preprint of P. Mitra and P. Roy (DESY 78/38) in which the operator solutions for massless QED_2 on the torus of $SU(N)_D$ are given. We disagree with their interpretation as a broken QCD_2 solution.

where

$$J^i_{5\mu}(x;\epsilon) = \bar{\psi}(x+\tfrac{1}{2}\epsilon)\gamma^5\gamma_\mu \tfrac{1}{2}\{\tfrac{1}{2}\lambda^i, 1 + ig\epsilon^\nu A_\nu(x)\}\psi(x-\tfrac{1}{2}\epsilon),$$
$$A_\mu = \sum \tfrac{1}{2}\lambda^i A^i_\mu, \tag{5.1}$$

and the symmetric limit is to be taken. It is easy to check that $J^i_{5\mu}(x;\epsilon)$ transforms covariantly under the gauge group.

Taking the divergence and making use of the equation of motion for the fermions, one obtains

$$\partial^\mu J^i_{5\mu}(x;\epsilon) = g\bar{\psi}(x+\tfrac{1}{2}\epsilon)\gamma^5\gamma^\mu[A_\mu(x), \tfrac{1}{2}\lambda^i]\psi(x-\tfrac{1}{2}\epsilon)$$
$$-\tfrac{1}{2}g\bar{\psi}(x+\tfrac{1}{2}\epsilon)\gamma^5\gamma^\mu\epsilon^\nu\{\tfrac{1}{2}\lambda^i, \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\}\psi(x-\tfrac{1}{2}\epsilon),$$

which can be written in the covariant form

$$\mathscr{D}^\mu_{ij} J^j_{5\mu}(x;\epsilon) = \tfrac{1}{2}ig\bar{\psi}(x+\tfrac{1}{2}\epsilon)\gamma^5\gamma^\mu\epsilon^\nu\{\tfrac{1}{2}\lambda^i, F_{\mu\nu}(x)\}\psi(x-\tfrac{1}{2}\epsilon), \tag{5.2}$$

where $\mathscr{D}^\mu_{ij}$ is the covariant derivative

$$\mathscr{D}^\mu_{ij} = \delta_{ij}\partial^\mu + f_{ikj}A^\mu_k .$$

For $\epsilon \to 0$ the only contribution comes from the (canonical short-distances singularity of the operator product of fermion fields so that

$$\mathscr{D}^\mu_{ij} J^j_{5\mu}(x;\epsilon) = \frac{g}{2\pi}\epsilon^{\mu\lambda}\frac{\epsilon_\lambda\epsilon^\nu}{\epsilon^2}F^i_{\mu\nu}(x) + \mathrm{O}(\epsilon). \tag{5.3}$$

It is convenient to write $F^i_{\mu\nu}$ in the form

$$F^i_{\mu\nu}(x) = -\frac{g}{\sqrt{2\pi}}\epsilon_{\mu\nu}\tilde{\Sigma}^i(x), \tag{5.4}$$

which in $1+1$ dimensions can always be done. From Maxwell's equations (assumed to be satisfied on a suitable subspace $\mathscr{H}_{\text{phys}}$ of $\mathscr{H}$

$$\mathscr{D}^\mu_{ij} F^j_{\mu\nu}(x) = -gJ^i_\nu(x),$$

one then has

$$J^i_\mu(x) = \frac{1}{\sqrt{2\pi}}\tilde{\mathscr{D}}^{ij}_\mu \tilde{\Sigma}^j \qquad \tilde{\mathscr{D}}_\mu = \epsilon_{\mu\nu}\mathscr{D}^\nu . \tag{5.5}$$

$\tilde{\Sigma}^i$ are evidently non-canonical (Lie fields) which play the role of the current potentials of (3.13) in the non-Abelian theory. Using (5.4) in (5.3) and taking the limit $\epsilon \to 0$ one arrives at the equation

$$\left(\mathscr{D}_\mu\mathscr{D}^\mu + \frac{g^2}{2\pi}\right)_{ij}\tilde{\Sigma}^j(x) = 0, \tag{5.6}$$

which is the covariant generalization of the torus equation (3.5) to SU(N). Hence the intrinsic Higgs mechanism already encountered in the Abelian generalization of QED_2 is also expected to be characteristic of QCD_2. As in the case of $SU(N)_D$ one again expects the mass term in eq. (5.6) to prevent the existence of colour sectors in $\mathscr{H}_{phys}$.

6. Conclusion

We have considered here some generalizations of QED_2 which are of particular interest and can be treated by methods analogous to those familiar from QED_2. In all cases "charge" sectors corresponding to quantum numbers to which the gauge fields couple are absent; this is a consequence of Gauss' law in two-dimensional space-time. It is *a priori* not clear whether this means screening of these quantum numbers or confinement of the particles carrying them. Thus, in (massless and massive) QED_2 the question of whether the operator $\hat{\psi}(x)$ creates a (zero-charge) state corresponding to a screened quark or a quark-antiquark bound state cannot be decided without developing a more detailed dynamical picture. Such a picture has been developed in ref. [17]. However, as has been brought out in the discussion of the previous sections, some clarification of the question of screening *versus* confinement can also be obtained independent of a more detailed dynamical investigation if additional quantum numbers are involved which do not couple to the gauge field and, hence, cannot be screened. The observed absence of states with these additional quantum numbers (fundamental flavour for SU(M) flavour and charge for $SU(N)_D$ colour) in the case of massive fermions (except for some special θ-worlds) indicates confinement rather than screening, in agreement with the semi-classical considerations of ref. [17].

We believe that our distinction between screening and confinement is also relevant for the recently discussed CP^{n-1} σ models with fermions [16]. The kink phenomenon leading to the appearance of exotic states for particular non-vanishing values of θ for massive fermions may also occur in those models.

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